~~convergent or divergent.~~ Use integral test to tell if con. or div.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

8. $\displaystyle\sum_{n=1}^{\infty} n^2e^{-n^3}$

3. The function is continuous, positive, and decreasing on $[1,\infty)$ so Integral Test applies

$$\int_1^{\infty} \frac{1}{\sqrt[5]{x}}\,dx = \int_1^{\infty} x^{-1/5}\,dx = \lim_{t\to\infty}\int_1^t x^{-1/5}dx$$

$$= \lim_{t\to\infty} \frac{5}{4}x^{4/5}\Big]_1^t$$

$$= \frac{5}{4}t^{4/5} - \frac{5}{4} = \frac{5}{4}(\infty)^{4/5} - 5/4$$

$$= \boxed{\infty \to \text{DIVERGES}}$$

* easier to prove DIVERGENCE w/ p-series test

4. The function is continuous, positive, + decreasing on $[1,\infty)$ so the integral test applies

$$\int_1^{\infty} \frac{1}{x^5}\,dx = \int_1^{\infty} x^{-5}\,dx = \lim_{t\to\infty}\int_1^t x^{-5}\,dx$$

$$= \lim_{t\to\infty} -\frac{1}{4}x^{-4}\Big]_1^t$$

$$= \frac{-1}{4t} - \frac{-1}{4(1)} = \frac{-1}{4t^4} + \frac{1}{4} = \frac{-1}{4\infty^4} + \frac{1}{4} = 0 + \frac{1}{4}$$

$$= \boxed{\frac{1}{4} \to \text{Converges}}$$

* easier to prove convergence w/ p-series test

–26 Determine whether the series is convergent or divergent.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$

10. $\displaystyle\sum_{n=3}^{\infty} n^{-0.9999}$

11. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots$

12. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots$

13. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots$

14. $\frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \frac{1}{17} + \cdots$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

18. $\displaystyle\sum_{n=3}^{\infty} \frac{3n-4}{n^2-2n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+6n+13}$

5. The function is continuous, positive, + decreasing on $[1,\infty)$, so the Integral test applies

$$\int_1^{\infty} \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(\underbrace{2x+1}_{u})^3}\,\underbrace{dx}_{\frac{1}{2}du} \leftarrow \begin{bmatrix} \text{let } u = 2x+1 \\ du = 2\,dx \\ \frac{1}{2}du = dx \end{bmatrix} = \frac{1}{2}\int_3^{2t+1} u^{-3}\,du$$

$$= \lim_{t\to\infty} -\frac{1}{4}u^{-2}\Big]_3^{2t+1}$$

$$= -\frac{1}{4}(2t+1)^{-2} - -\frac{1}{4}(3)^{-2} = -\frac{1}{4}\cdot\frac{1}{\sqrt{2t+1}} + \frac{1}{36} = \frac{-1}{4\sqrt{2\infty+1}} + \frac{1}{36} = 0 + \frac{1}{36} =$$

$$= \boxed{\frac{1}{36} \to \text{converges}}$$

6. The function is continuous, positive, and decreasing on $[1,\infty)$ so the integral test can be used

$$\int_1^\infty \frac{1}{\sqrt{x+4}}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{x+4}}\,dx = \int_1^t (x+4)^{-1/2}\,dx$$

$$= \lim_{t\to\infty} 2\sqrt{x+4}\,\Big]_1^t = 2\sqrt{t+4} - 2\sqrt{1+4} = 2\sqrt{\infty+4} - 2\sqrt{5} = \infty$$

$$= \boxed{\infty \to \text{DIVERGES}}$$

7. The function is continuos, positive, and decreasing on $[1,\infty)$ so the integral test can be used

$$\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{t\to\infty}\int_1^t \frac{\overbrace{x\,dx}^{\frac{1}{2}du}}{\underbrace{x^2+1}_{u}} \leftarrow \begin{bmatrix} \text{let } u = x^2+1 \\ du = 2x\,dx \\ \frac{1}{2}du = x\,dx \end{bmatrix} = \frac{1}{2}\int_2^{t^2+1} \frac{du}{u}$$

$$= \lim_{t\to\infty} \frac{1}{2}\ln|u|\,\Big]_2^{t^2+1} = \frac{1}{2}\ln|t^2+1| - \frac{1}{2}\ln 2 = \frac{1}{2}\ln|t^2+1| - \ln\sqrt{2} = \frac{1}{2}\ln\infty - \ln\sqrt{2}$$

$$= \boxed{\infty \to \text{DIVERGES}}$$

8. The function is continuous, positive, and decreasing on $[1,\infty)$ so the integral test applies

$$\int_1^\infty x^2\cdot e^{-x^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{\overbrace{x^2\,dx}^{\frac{1}{3}du}}{e^{\underbrace{x^3}_{u}}} \quad \begin{bmatrix} \text{let } u = x^3 \\ du = 3x^2\,dx \\ \frac{1}{3}du = x^2\,dx \end{bmatrix} = \frac{1}{3}\int_1^{t^3} e^{-u}\,du$$

$$= \lim_{t\to\infty} -\frac{1}{3}e^{-u}\,\Big]_1^{t^3} = -\frac{1}{3}e^{-t^3} - -\frac{1}{3}e^{-1} = \frac{-1}{3e^{t^3}} + \frac{1}{3e} = \frac{-1}{3e^\infty} + \frac{1}{3e} = 0 + \frac{1}{3e}$$

$$= \boxed{\frac{1}{3e} \to \cancel{\text{DIVERGES}}\ \text{CONVERGES}}$$

Determine whether series is convergent or divergent.

9. $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}}$ is a p-series + converges since $p = \sqrt{2} > 1$

10. $\boxed{\sum_{n=3}^{\infty} n^{-0.9999}}$ $= \sum_{n=3}^{\infty} \frac{1}{n^{0.9999}}$ is a p-series + diverges since $p = 0.9999 \leq 1$

11. $\boxed{1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \ldots}$

$= \sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series + converges since $p = 3 > 1$

12. $\boxed{1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \ldots}$

$= \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n \cdot n^{1/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a p-series + converges since $p = 3/2 > 1$

13. $\boxed{1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \ldots}$

$= \sum_{n=1}^{\infty} \frac{1}{2n-1}$ } The function $f(x) = \frac{1}{2x-1}$ is continuous, positive, and decreasing on $[1, \infty)$ so integral test applies

$$\int_1^{\infty} \frac{1}{2x-1}\,dx = \lim_{t \to \infty} \int_1^t \frac{1}{2x-1} = \frac{1}{2} \ln|2x-1| \Big]_1^t$$

$$= \frac{1}{2} \ln|2t-1| - \cancel{\frac{1}{2} \ln|2-1|}^{\,0}$$

$$= \frac{1}{2} \ln(2\infty - 1) = \boxed{\infty \rightarrow \text{DIVERGES}}$$

14. $\boxed{\frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \frac{1}{17} + \dots}$

$$= \sum_{n=1}^{\infty} \frac{1}{3n+2}$$

The function $f(x) = \frac{1}{3x-2}$ is continuous, positive, and decreasing on $[1, \infty)$ so integral test applies

$$\int_1^{\infty} \frac{1}{3x-2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{3x+2}\,dx = \frac{1}{3}\ln|3x+2|\Big]_1^t$$

$$= \frac{1}{3}\ln|3t+2| - \frac{1}{3}\ln|3+2|$$

$$= \frac{1}{3}\ln|3\infty+2| - \frac{1}{3}\ln 5 = \boxed{\infty \to \text{DIVERGES}}$$

15. $\boxed{\sum_{n=1}^{\infty} \frac{\sqrt{n}+4}{n^2}}$

$$= \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

p-series with $p = 3/2 > 1$

multiple of p-series w/ $p = 2 > 1$

CONVERGES

16. $\boxed{\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}}$

The function $f(x) = \frac{x^2}{x^3+1}$ is continuous, positive, and decreasing on $[1, \infty)$ so integral test applies

$$\int_1^{\infty} \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{x^2}{x^3+1}\,dx$$

let $u = x^3+1$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$$= \frac{1}{3}\int_1^t \frac{du}{u}$$

$$= \frac{1}{3}\ln|u|\Big]_1^t$$

$$= \frac{1}{3}\ln|t| - \cancel{\ln 1}^{\,0} = \frac{1}{3}\ln\infty = \infty$$

$$= \boxed{\infty \to \text{DIVERGES}}$$

17. $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2+4}}$

The function $f(x) = \frac{1}{x^2+4}$ is continuous, positive, and decreasing on $[1, \infty)$ so integral test applies

$$\int_1^{\infty} \frac{1}{x^2+4}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+4}\,dx = \int_1^t \frac{1}{\frac{4x^2}{4}+4}\,dx = \int_1^t \frac{1}{4\left(\frac{x^2}{4}+1\right)}\,dx$$

$$= \frac{1}{4}\int_1^t \frac{1}{1+\frac{x^2}{4}}\,dx = \frac{1}{4}\int_1^t \frac{1}{1+\left(\frac{x}{2}\right)^2}\,dx$$

$$= \frac{1}{4}\cdot\arctan\left(\frac{x}{2}\right)\cdot 2\Big]_1^t = \frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big]_1^t$$

$$= \frac{1}{2}\arctan\left(\frac{t}{2}\right) - \frac{1}{2}\arctan\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\cdot\pi/2 - \frac{1}{2}\cdot\arctan(1/2) = \frac{\pi}{4} - \frac{1}{2}\arctan(1/2) = \boxed{\text{a constant} \to \text{converges}}$$

18. $\boxed{\sum_{n=3}^{\infty} \frac{3n-4}{n^2-2n}}$ } The function $f(x) = \frac{3x-4}{x^2-2x}$ is continuous, positive, and decreasing on $[3,\infty)$ so Integral Test applies

$$\int_3^{\infty} \frac{3x-4}{x^2-2x}\,dx = \lim_{t\to\infty} \int_3^t \frac{3x-4}{x^2-2x}\,dx$$ [note that u-sub would not help simplify this function : $\cancel{u = x^2-2x}$, $\cancel{du = 2x-2}$]

PARTIAL FRACTIONS!!

$$\frac{3x-4}{x^2-2x} = \frac{3x-4}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

$$\begin{aligned} 3x-4 &= A(x-2) + B(x) \\ &= Ax - 2A + Bx \\ &= \underbrace{(A+B)}_{3}x - \underbrace{2A}_{-4} \end{aligned}$$

$\begin{bmatrix} A+B = 3 \\ -2A = -4 \end{bmatrix}$ systems of eqs.

so $A = 2$ + $B = 1$

$$\lim_{t\to\infty} \int_3^t \left(\frac{2}{x} + \frac{1}{x-2}\right)dx = 2\ln|x| + \ln|x-2| \Big]_3^t$$

$$= 2\ln|t| + \ln|t-2| - [2\ln 3 + \cancel{\ln 1}^{0}]$$

$$= 2\ln\infty + \cancel{\ln 1}^{0} + \ln|\infty - 2| - 2\ln 3 =$$

$$= \boxed{\infty \to \text{DIVERGES}}$$

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19. $\boxed{\sum_{n=1}^{\infty} \frac{\ln n}{n^3}}$ } The function $f(x) = \frac{\ln x}{x^3}$ is continuous, positive, and *decreasing* on $[1,\infty)$ so Integral Test applies

$$\int_1^{\infty} \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^3}\,dx$$ [note again that u-sub would not help simplify this integral]

INTEGRATION BY PARTS!! $\left[\int u\,dv = uv - \int v\,du\right]$

$u = \ln x$ $\qquad v = -\frac{1}{2}x^{-2}$

$du = \frac{1}{x}$ $\qquad dv = x^{-3}\,dx$

$$= \ln x \cdot \frac{-1}{2x^2} - \int_1^t -\frac{1}{2}x^{-2} \cdot \frac{1}{x}\,dx = \frac{-\ln x}{2x^2} + \frac{1}{2}\int_1^t x^{-3}\,dx$$

continues next page ⟶

$$= \frac{-\ln x}{2\cdot x^2} + \frac{1}{2}\int_2^t x^{-3}\,dx = \frac{-\ln x}{2x^2} - \frac{1}{4}x^{-2}\Big]_2^t$$

$$= \left[\frac{-\ln t}{2t^2} - \frac{1}{4t^2}\right] - \left[\frac{\cancel{-\ln 2}^{\,0}}{\cancel{2(2)^2}} - \frac{1}{4\cdot 2^2}\right]$$

$$= \frac{-\ln t}{2t^2} - \frac{1}{4t^2} + \frac{1}{8} + \frac{-\ln 2}{8}$$

$$\lim_{t\to\infty} \frac{-\ln t}{2t^2} - 0 + \frac{1}{8} + \frac{-\ln 2}{8}$$

L'Hospitals

$$\frac{-\frac{1}{t}}{4t} = \frac{-1}{4t^2} = \frac{-1}{4\infty^2} = 0$$

$$\frac{1}{8} + \frac{-\ln 2}{8} + 0 = \frac{1}{8} - \frac{\ln 2}{8}$$

$\boxed{\text{converges}}$

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20. $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2+6n+13}}$ the function $f(x)$ is positive, decreasing, and continuous on $[1,\infty)$ so Integral Test applies

$$\int_1^{\infty} \frac{1}{x^2+6x+13}\,dx \overset{\text{*see side work}}{=} \lim_{t\to\infty}\int_1^t \frac{1}{(x+3)^2+4}\,dx$$

similar to $\int_1^t \frac{1}{x^2+1}\,dx = \arctan x\Big]_1^t$

side work: completing the square.

$x^2+6x+13=0$

$x^2+6x+9=-13+9$

$(x+6)^2=-4$

$(x+6)^2+4$

$$= \lim_{t\to\infty}\int_1^t \frac{1}{4\left(\frac{1}{4}(x+3)^2+1\right)}\,dx \overset{\lim_{t\to\infty}}{=} \frac{1}{4}\int_1^t \frac{1\,dx}{\left(\frac{x+3}{2}\right)^2+1}$$

$u=\left(\frac{x+3}{2}\right)$

$du=\frac{1}{2}\,dx$

$2\,du=dx$

$$\frac{1}{4}\int_2^{\frac{t+3}{2}} \frac{1\cdot 2\,du}{u^2+1} = \frac{1}{2}\int_2^{\frac{t+3}{2}} \frac{1}{u^2+1}\,du$$

$$= \frac{1}{2}\arctan u\Big]_2^{\frac{t+3}{2}} = \frac{1}{2}\arctan\left(\frac{t+3}{2}\right) - \frac{1}{2}\arctan 2$$

$$= \frac{1}{2}\cdot \pi/2 - \frac{1}{2}\arctan 2 \rightarrow \boxed{\text{converges}}$$