1a. The series $\sum a_n$ ~~absolutely converges~~

b. The series $\sum a_n$ absolutely converges

c. The series $\sum a_n$

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2. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

$$\lim_{n\to\infty} \left| \frac{\frac{(-2)^{n+1}}{(n+1)^2}}{\frac{(-2)^n}{n^2}} \right| = \left| \frac{(-2)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-2)^n} \right| = \frac{\cancel{(-2)^n} \cdot 2 \cdot n^2}{(n+1)^2 \cdot \cancel{(-2)^n}}$$

$$= \frac{2n^2}{(n+1)^2} \xrightarrow{\text{L'Hospital's}} \frac{4n}{2(n+1)} \to \frac{4}{2} = \boxed{2}$$

By ratio test, series is ~~absolutely convergent~~ divergent

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3. $\sum_{n=1}^{\infty} \frac{n}{5^n}$

$$\lim_{n\to\infty} \left| \frac{\frac{n+1}{5^{n+1}}}{\frac{n}{5^n}} \right| = \frac{n+1}{5^{n+1}} \cdot \frac{5^n}{n} = \frac{n+1 \cdot \cancel{5^n}}{\cancel{5^n} \cdot 5 \cdot n} = \frac{n+1}{5n} \to \boxed{\frac{1}{5}}$$

By ratio test, series is absolutely convergent

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4. $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{n}{n^2+4}$

$$\lim_{n\to\infty} \left| \frac{\frac{n+1}{(n+1)^2+4}}{\frac{n}{n^2+4}} \right| = \frac{n+1}{(n+1)^2+4} \cdot \frac{n^2+4}{n} \overset{\text{L'Hospitals}}{=} \frac{n^3+n^2+4n+5}{n^3+2n^2+5n} \to \boxed{1}$$

By ratio test, inconclusive! Try another test:

Alternating series test:

① $\frac{n+1}{(n+1)^2+4} \le \frac{n}{n^2+4}$ for all n for $n \ge 2$

② $\lim_{n\to\infty} \frac{n}{n^2+4} \to \frac{1}{2n} =$

$\boxed{\text{Converges}}$ by Alt. series test

what about absolute convergenc? ...

if $\sum |a_n|$ converges, then $\sum a_n$ converges (absolutely)

$$\sum \left| (-1)^{n-1} \cdot \frac{n}{n^2+4} \right| = \sum \frac{n}{n^2+4}$$

integral test ~~$\frac{n}{n^2+4}$~~ $y = \frac{x}{x^2+4}$ is pos, cont. + dec. on $[1, \infty)$

$$\lim_{t\to\infty} \int_1^t \frac{x}{x^2+4}\,dx = \frac{1}{2} \ln|2x+4| \Big]_1^t = \infty$$

DIVERGES

So original seres is $\boxed{\text{conditionally convergent}}$

5. $\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}$

$\lim_{n\to\infty} \left| \frac{\frac{1}{5(n+1)+1}}{\frac{1}{5n+1}} \right| = \frac{1}{5(n+1)+1} \cdot \frac{5n+1}{1} = \frac{5n+1}{5n+6} \xrightarrow{\text{L'Hospitals}} \frac{5}{5} = \boxed{1}$

By ratio test, $\boxed{\text{inconclusive}}$! Try another test

Alt. series test:

① $\frac{1}{5(n+1)+1} < \frac{1}{5n+1}$

$\frac{1}{5n+6} < \frac{1}{5n+1}$ for all n ✓

② $\lim_{n\to\infty} \frac{1}{5n+1} = 0$ ✓

$\boxed{\text{Converges}}$ by Alt. series test

check for absolute convergence:

$\lim_{t\to\infty} \int_0^t \frac{1}{5x+1}\,dx = \frac{1}{5}\ln|5x+1| \Big]_0^t$

$= \frac{1}{5}\ln|\infty| =$ DIVERGES

So original series $\boxed{\text{Conditionally converges}}$

6. $\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$

$$\lim_{n\to\infty} \left| \frac{\frac{(-3)^{n+1}}{(2(n+1)+1)!}}{\frac{(-3)^n}{(2n+1)!}} \right| = \left| \frac{\cancel{(-3)^n} \cdot -3}{(2n+3)!} \cdot \frac{(2n+1)!}{\cancel{(-3)^n}} \right| = \frac{3 \cdot \cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}}$$

$$= \frac{3}{(2n+3)(2n+2} = \frac{3}{4n^2+\dots} \to \boxed{0}$$

By ratio test, series is absolutely convergent

7. $\sum_{K=1}^{\infty} K\left(\frac{2}{3}\right)^K$

$$\lim_{n\to\infty} \left| \frac{(K+1)\left(\frac{2}{3}\right)^{K+1}}{K\left(\frac{2}{3}\right)^K} \right| = \frac{(K+1)\cancel{\left(\frac{2}{3}\right)^K} \cdot \frac{2}{3}}{K\cancel{\left(\frac{2}{3}\right)^K}} = \frac{\frac{2}{3}(K+1)}{K} = \frac{\frac{2}{3}\cancel{K}}{\cancel{K}} + \frac{\frac{2}{3}}{K}$$

$$= \frac{2}{3} + \frac{2}{3K} = \frac{2}{3} + 0 = \frac{2}{3}$$

By ratio test, series is absolutely convergent

8. $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}} \right| = \frac{(n+1)!}{100^{n+1}} \cdot \frac{100^n}{n!} = \frac{(n+1)\cdot \cancel{n!} \cdot \cancel{100^n}}{\cancel{100^n} \cdot 100 \cdot \cancel{n!}}$$

$$= \frac{n+1}{100} = \infty$$

By ratio test, series $\boxed{\text{diverges}}$

9. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{(1.1)^n}{n^4}$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} \cdot (1.1)^{n+1}}{(n+1)^4} \right| = \frac{\frac{(1.1)^{n+1}}{(n+1)^4}}{\frac{(1.1)^n}{n^4}} = \frac{\cancel{(1.1)^n} \cdot (1.1) \cdot n^4}{(n+1)^4 \cdot \cancel{(1.1)^n}} = \frac{(1.1)n^4}{(n+1)^4} \to 1.1$$

By ratio test, $\boxed{\text{series diverges}}$

10. Alt. series test:

$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{\sqrt{n^3+2}}$

1. $\frac{n+1}{\sqrt{(n+1)^3+2}} \leq \frac{n}{\sqrt{n^3+2}}$ ✓

2. $\lim_{n\to\infty} \frac{n}{\sqrt{n^3+2}} = 0$ ✓

By Alt. series test, series converges