$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

Maclaurin series

Maclaurin:

5. $\boxed{f(x) = (1-x)^{-2}}$

n	$f^n(x)$	$f^n(0)$	$\frac{f^n(0)}{n!}$
0	$(1-x)^{-2}$	1	1
1	$2(1-x)^{-3}$	2	2
2	$6\cancel{24}(1-x)^{-4}$	6	3
3	$24(1-x)^{-5}$	24	4
4	$120(1-x)^{-6}$	120	5

note pattern: $(n+1)$

not necessary but can be helpful.

write out first few terms!

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x + \ldots$$

$$= 1 + 2x + \frac{6}{2}x^2 + \frac{24}{6}x^3 + \ldots$$

$$= 1 + 2x + 3x^2 + 4x^3 + \ldots$$

$$\boxed{= \sum_{n=0}^{\infty} (n+1)x^n}$$

Radius of convergence:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+2)\cancel{x^n}\cdot x}{(n+1)\cdot\cancel{x^n}}\right| = |x|$$

converges when $|x| < 1$, so $\boxed{\text{radius of con.} = 1}$

6. Maclaurin

$$\boxed{f(x) = \ln(1+x)}$$

n	$f^n(x)$	$f^n(0)$	$\frac{f^n(0)}{n!}$
0	$\ln(1+x)$	0	0
1	$\frac{1}{1+x}$	1	1
2	$-\frac{1}{(x+2)^2}$	-1	$-1/2$
3	$\frac{2}{(1+x)^3}$	2	$1/3$
4	$\frac{-6}{(1+x)^4}$	-6	$-1/4$

notice pattern: $(-1)^{n+1} \cdot \frac{1}{n}$

not necessary but can be helpful

write out first few terms:

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

$$= 0 + x - \frac{1}{2}x^2 + \frac{2}{6}x^3 - \frac{6}{24}x^4 + \dots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$= \boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n} \cdot x^n}$$

Radius of convergence:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{n+1} \cdot \cancel{x^n} \cdot x \cdot \frac{n}{\cancel{x^n}}\right| = |x|$$

converges when $|x| < 1$, so $\boxed{\text{radius of con: } 1}$

maclaurin

7. $\boxed{f(x) = \sin \pi x}$

n	$f^n(x)$	$f^n(0)$	$\frac{f^n(0)}{n!}$
0	$\sin \pi x$	0	0
1	$\pi \cos \pi x$	π	π
2	$-\pi^2 \sin \pi x$	0	0
3	$-\pi^3 \cos \pi x$	$-\pi^3$	$\frac{-\pi^3}{6}$

notice pattern!

$(-1)^n \cdot \frac{\pi^{2n+1}}{(2n+1)!}$

not necessary but can be helpful

write out first few terms:

$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$

$= 0 + \pi x + 0 - \frac{\pi^3}{3!}x^3 + 0 + \frac{\pi^5}{5!}x^5 + \dots$

$= \pi x - \frac{\pi^3}{3!}x^3 + \frac{\pi^5}{5!}x^5 - \frac{\pi^7}{7!}x^7 + \dots$

$= \boxed{\sum_{n=0}^{\infty} (-1)^n \cdot \frac{\pi^{2n+1}}{(2n+1)!} \cdot x^{2n+1}}$

Radius of convergence:

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{\pi^{2n+3} \cdot x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1} \cdot x^{2n+1}} \right| = \frac{\pi^2 x^2}{(2n+3)(2n+2)}$

$= 0 < 1$

for all x

so $\boxed{\text{radius of con is } \infty}$

Taylor

13. $\boxed{f(x) = x^4 - 3x^2 + 1, \quad a = 1}$

$$F(x) = \sum_{n=0}^{\infty} \frac{f^n(1)}{n!}(x-1)^n$$

$n=0$	$f^n(x)$	$f^n(1)$	$\frac{f^n(1)}{n!}$
0	x^4-3x^2+1	-1	-1
1	$4x^3-6x$	-2	-2
2	$12x^2-6$	6	3
3	$24x$	24	4
4	24	24	1
5	0	0	0

$\downarrow \quad \downarrow$

not necessary but can be helpful

write out first few terms:

$$= \frac{-1}{0!}(x-1)^0 + \frac{-2}{1!}(x-1)^1 + \frac{6}{2!}(x-1)^2 + \frac{24}{3!}(x-1)^3 + \ldots$$

$$= \boxed{-1 - 2(x-1) + 3(x-1)^2 + 4(x-1)^3 + (x-1)^4}$$

$=$

* note *

no $\sum_{n=0}^{\infty}$ required since this is not an infinite expansion!... this is a FINITE series

a finite series converges for all x

so $\boxed{\text{radius of con.} = \infty}$

Taylor

14. $\boxed{f(x) = x - x^3, \quad a = -2}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(-2)}{n!}(x-a)^n$$

n	$f^n(x)$	$f^n(-2)$	$\frac{f^n(-2)}{n!}$
0	$x - x^3$	6	6
1	$1 - 3x^2$	-11	-11
2	$-6x$	-12	-6
3	-6	-6	-1
4	0	0	0
↓	↓	↓	↓

first few terms:

$= \frac{6}{0!}(x+2)^0 + \frac{-11}{1!}(x+2)^1 + \frac{12}{2!}(x+2)^2 + \frac{-6}{3!}(x+2)^3$

$= \boxed{6 - 11(x-2) + 6(x+2)^2 - (x+2)^3}$

* note *

no $\sum_{n=0}^{\infty}$ required since this is NOT an infinite series... this is FINITE

a finite series converges for all x

so $\boxed{\text{radius of con.} = \infty}$

Taylor

15. $\boxed{f(x) = \ln x, \quad a = 2}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(2)}{n!}(x-2)^n$$

n	$f^n(x)$	$f^n(2)$	$\frac{f^n(2)}{n!}$
0	$\ln x$	$\ln 2$	$\ln 2$
1	$1/x$	$1/2$	$1/2$
2	$-1/x^2$	$\frac{-1}{2^2} = -\frac{1}{4}$	$-1/8$
3	$2/x^3$	$\frac{2}{2^3} = \frac{1}{8}$	$1/48$
4	$-6/x^4$	$\frac{-6}{2^4} = \frac{-6}{16}$	$-1/16$

notice pattern after $n=0$: $(-1)^{n+1}\frac{1}{n2^n}$

not necessary but can be helpful

write out first few terms:

$$= \frac{\ln 2}{0!}(x-2)^0 + \frac{1}{1!\,2^1}(x-2)^1 + \frac{-1}{2!\,4}(x-2)^2 + \frac{2}{3!\,8}(x-2)^3 + \ldots$$

$$= \ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n-1)!}{n!\,2^n}(x-2)^n$$

OR

$$= \ln 2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n\,2^n}(x-2)^n$$

Radius of convergence:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cancel{(x-2)^n}(x-2)}{(n+1)\cdot\cancel{2^n}\cdot 2}\cdot\frac{n\cdot\cancel{2^n}}{\cancel{(x-2)^n}}\right| = \frac{|x-2|}{2} < 1$$

$$|x-2| < 2$$

$\boxed{\text{Radius of con.} = 2}$