Power Series Test Review

| | Maclaurin Series | First three terms |
|---|---|---|
| $\sin x$ | $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$ | $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ |
| $\cos x$ | $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$ | $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ |
| $e^x$ | $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ | $1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$ |
| $\frac{1}{1-x}$ | $\sum_{n=0}^{\infty} x^n$ | $1 + x + x^2 + \dots$ |

Obtain the Maclaurin series for the given function.

1. $f(x) = \sin \pi x$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot \pi^{2n+1} \cdot x^{2n+1}}{(2n+1)!}$$

2. $f(x) = \cos \frac{\pi x}{2}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot (\pi/2)^{2n} \cdot x^{2n}}{(2n)!}$$

3. $f(x) = e^x + e^{2x}$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} + \frac{(2x)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} [1 + 2^n]$$

no calculator needed →

4. Use the Maclaurin series for $e^x$ to calculate $1/\sqrt[10]{e}$ correct to five decimal places.

$$\frac{1}{\sqrt[10]{e}} = \frac{1}{e^{1/10}} = e^{-1/10}$$

if $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

then $e^{-1/10} = 1 + \frac{-1/10}{1!} + \frac{(-1/10)^2}{2!} + \frac{(-1/10)^3}{3!} + \frac{(-1/10)^4}{4!}$

$$= 1 - \frac{1}{10} + \frac{1}{200} - \frac{1}{6000} + \frac{1}{240000}$$

note that the next term would not affect the first 5 decimal places

ACTUAL: $e^{-1/10} = 0.904837418$

5. a. Expand $1/\sqrt[4]{1+x}$ as a power series.

| $n$ | $f^n(x)$ | $f^n(0)$ |
|---|---|---|
| 0 | $(x+1)^{-1/4}$ | 1 |
| 1 | $-\frac{1}{4}(x+1)^{-5/4}$ | $-1/4$ |
| 2 | $5/16(x+1)^{-9/4}$ | $5/16$ |
| 3 | $-\frac{45}{64}(x+1)^{-13/4}$ | $-45/64$ |

$$= \frac{1}{0!}x^0 + \frac{-1/4}{1!}x + \frac{5/16}{2!}x^2 + \frac{-45/64}{3!}x^3$$

$$= 1 - \frac{1}{4}x + \frac{5}{32}x^2 - \frac{45}{384}x^3 + \dots$$

not needed →

b. Use part (a) to estimate $1/\sqrt[4]{1.1}$ correct to three decimal places.

note that if $x = 0.1$, then: $\frac{1}{\sqrt[4]{1+x}} = \frac{1}{\sqrt[4]{1.1}}$

$$\frac{1}{\sqrt[4]{1.1}} = 1 - \frac{1}{4}(0.1) + \frac{5}{32}(0.1)^2$$

← note that the next term would be $-\frac{45}{384}(0.1)^3$ which would not affect first three decimal places & is not needed

Find the radius of convergence and interval of convergence of the series.

6. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$

$\lim_{n\to\infty} \left| \frac{x^{n+1}}{\sqrt[3]{n+1}} \cdot \frac{\sqrt[3]{n}}{x^n} \right| = \left| x \cdot \frac{\sqrt[3]{n}}{\sqrt[3]{n+1}} \right| = |x| < 1$

radius of con = 1

check endpts $x = \pm 1$:

if $x = -1$, then $\sum \frac{(-1)^n (-1)^n}{\sqrt[3]{n}} = \sum \frac{1}{\sqrt[3]{n}}$ X

if $x = 1$, then $\sum \frac{(-1)^n \cdot 1^n}{\sqrt[3]{n}} = \sum \frac{(-1)^n}{\sqrt[3]{n}}$ ✓

interval of convergence: $-1 < x \le 1$

7. $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$

$\lim_{n\to\infty} \left| \frac{\cancel{10^n} \cdot 10 \cdot \cancel{x^n} \cdot x}{(n+1)^3} \cdot \frac{n^3}{\cancel{10^n x^n}} \right|$

$= |10x| < 1$

$|x| < 1/10$

radius of con $= \frac{1}{10}$

check endpts $x = \pm 1/10$

if $x = -1/10$, then $\sum \frac{10^n \cdot (-\frac{1}{10})^n}{n^3} = \sum \frac{(-1)^n}{n^3}$ ✓

if $x = 1/10$, then $\sum \frac{10^n \cdot (\frac{1}{10})^n}{n^3} = \sum \frac{(1)^n}{n^3} = \sum \frac{1}{n^3}$ ✓

interval $= [-1/10, 1/10]$

Find a power series representation for the function and determine the interval of convergence.

* #8-9 are geometric *

8. $f(x) = \frac{5}{1-4x^2} = 5 \cdot \frac{1}{1-(4x^2)}$ r

$= \sum_{n=0}^{\infty} 5 \cdot (4x^2)^n$

$= \sum_{n=0}^{\infty} 5 \cdot 4^n \cdot x^{2n}$

converges if $|4x^2| < 1 \rightarrow |x| < 1/2$

Radius of convergence is $\frac{1}{2}$.

~~check both endpts $x = \pm 1/2$ for convergence~~

no need to check endpts since geometric

interval of con: $(-1/2, 1/2)$

9. $f(x) = \frac{1+x}{1-x} = 1+x \cdot \frac{1}{1-(x)}$ r

$= \sum_{n=0}^{\infty} (1+x) \cdot x^n$

converges if $|x| < 1$

Radius of convergence is 1

~~check both endpts $x = \pm 1$ for con.~~

interval of convergence: $-1 < x < 1$

10. Let $P(x) = 3x^2 - 5x^3 + 7x^4 + 3x^5$ be the fifth-degree Taylor polynomial for the function $f$ about x=0. What is the value of $f'''(0)$?

$f' = P' = 6x - 15x^2 + 28x^3 + 15x^4$

$f'' = P'' = 6 - 30x + 84x^2 + 60x^3$

$f''' = P''' = -30 + 168x + 180x^2$

$\boxed{f'''(0) = P'''(0) = -30}$