WORKSHEET: DEFINITION OF THE DERIVATIVE

1. For each function given below, calculate the **derivative at a point** $f'(a)$ using the limit definition.

(a) $f(x) = 2x^2 - 3x$ $f'(0) = ?$

(b) $f(x) = \sqrt{2x+1}$ $f'(4) = ?$

(c) $f(x) = \dfrac{1}{x-2}$ $f'(3) = ?$

1a. $$\lim_{h\to 0} \frac{\boxed{f(0+h)} - \boxed{f(0)}}{h} = \frac{\boxed{2(0+h)^2 - 3(0+h)} - \boxed{0}}{h}$$

$$= \frac{2h^2 - 3h}{h} = \frac{\cancel{h}(2h-3)}{\cancel{h}} = 2h - 3$$

$$= 2(0) - 3 = 0 - 3 = \boxed{-3}$$

1b. $$\lim_{h\to 0} \frac{\boxed{f(4+h)} - \boxed{f(4)}}{h} = \frac{\boxed{\sqrt{2(4+h)+1}} - \boxed{3}}{h}$$

$$= \frac{\sqrt{9+2h} - 3}{h} \cdot \frac{\sqrt{9+2h}+3}{\sqrt{9+2h}+3} = \frac{\cancel{9} + 2\cancel{h} - \cancel{9}}{\cancel{h}(\sqrt{9+2h}+3)}$$

$$= \frac{2}{\sqrt{9+2h}+3} = \frac{2}{\sqrt{9+2(0)}+3} = \frac{2}{\sqrt{9}+3} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

1c. $$\lim_{h\to 0} \frac{\boxed{f(3+h)} - \boxed{f(3)}}{h} = \frac{\boxed{\frac{1}{(3+h)-2}} - \boxed{1}}{h} = \frac{\frac{1}{1+h} - 1}{h}$$

$$= \frac{\frac{1}{1+h} - \frac{1+h}{1+h}}{h} = \frac{\frac{\cancel{1}-\cancel{1}-h}{1+h}}{h} = \frac{-\cancel{h}}{1+h} \cdot \frac{1}{\cancel{h}} = \frac{-1}{1+0} = \boxed{-1}$$

1. (a) $f'(0) = -3$ (b) $f'(4) = 1/3$ (c) $f'(3) = -1$

2. For each function $f(x)$ given below, find the **general derivative** $f'(x)$ as a new function by using the limit definition.

(a) $f(x) = \sqrt{x-4}$ $f'(x) = ?$

(b) $f(x) = -x^3$ $f'(x) = ?$

(c) $f(x) = \dfrac{x}{x+1}$ $f'(x) = ?$

(d) $f(x) = \dfrac{1}{\sqrt{x}}$ $f'(x) = ?$

2a. $$\lim_{h\to 0} \frac{\boxed{f(x+h)} - \boxed{f(x)}}{h} = \frac{\boxed{\sqrt{x+h-4}} - \boxed{\sqrt{x-4}}}{h} \cdot \frac{\sqrt{x+h-4}+\sqrt{x-4}}{\sqrt{x+h-4}+\sqrt{x-4}}$$

$$= \frac{\cancel{x} + \cancel{h} - \cancel{4} - \cancel{x} + \cancel{4}}{\cancel{h}(\sqrt{x+h-4}+\sqrt{x-4})} = \frac{1}{\sqrt{x+h-4}+\sqrt{x-4}} = \frac{1}{\sqrt{x-4}+\sqrt{x-4}} = \boxed{\frac{1}{2\sqrt{x-4}}}$$

2b. $$\lim_{h\to 0} \frac{\boxed{f(x+h)} - \boxed{f(x)}}{h} = \frac{\boxed{-(x+h)^3} - \boxed{(-x^3)}}{h}$$

$$= \frac{-(x^3 + 2x^2h + xh^2 + hx^2 + 2xh^2 + h^3) + x^3}{h}$$

$$= \frac{-\cancel{x^3} - 2x^2h - xh^2 - hx^2 - 2xh^2 - h^3 + \cancel{x^3}}{h} = \frac{\cancel{h}(-2x^2 - xh - x^2 - 2xh - h^2)}{\cancel{h}}$$

$$= -2x^2 - \cancel{x(0)} - x^2 - \cancel{2x(0)} - h^2 = \boxed{-3x^2}$$

2c. $$\lim_{h\to 0} \frac{\boxed{f(x+h)} - \boxed{f(x)}}{h} = \frac{\frac{x+1}{x+1} \cdot \boxed{\frac{x+h}{x+h+1}} - \boxed{\frac{x}{x+1}} \cdot \frac{x+h+1}{x+h+1}}{h} = \frac{\frac{(x+1)(x+h) - x(x+h+1)}{(x+1)(x+h+1)}}{h}$$

$$= \frac{\cancel{x^2} + \cancel{xh} + \cancel{x} + h - \cancel{x^2} - \cancel{xh} - \cancel{x}}{(x+1)(x+h+1)} \cdot \frac{1}{\cancel{h}} = \frac{1}{(x+1)(x+h+1)} = \boxed{\frac{1}{(x+1)^2}}$$

2d. $$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \cdot \frac{\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}}$$

$$= \frac{\frac{x}{x} \cdot \frac{1}{x+h} - \frac{1}{x} \cdot \frac{x+h}{x+h}}{h\left(\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}\right)} = \frac{\cancel{x} - \cancel{x} - \cancel{h}}{x(x+h)} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\cancel{h}}$$

$$= \frac{-1(\sqrt{x+h}+\sqrt{x})}{x(x+h)} = \frac{-1(\sqrt{x+0}+\sqrt{x})}{x(x+0)} = \frac{-\sqrt{x}-\sqrt{x}}{x^2}$$

$$= \boxed{\frac{-2\sqrt{x}}{x^2}} = \frac{-2x^{1/2}}{x^2} = \frac{-2}{x^2 \cdot x^{-1/2}} = \boxed{\frac{-2}{x^{3/2}}}$$

2. (a) $f'(x) = \frac{1}{2\sqrt{x-4}}$ (b) $f'(x) = -3x^2$ (c) $f'(x) = \frac{1}{(x+1)^2}$ (d) $f'(x) = \frac{-1}{2x^{3/2}}$